Journal of Statistical Physics, Vol. 45, Nos. 1/2, 1986

Relaxation Problem with Quadratic Noise

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Received March 6, 1986

A linear first-order equation with a quadratic colored noise is considered. An exact one-dimensional probability distribution of the process is obtained from the characteristic function. The characteristic function is calculated by means of special functionals of the noise. An auxiliary set of three ordinary differential equations (which contains a Riccati equation) is solved for all values of parameters of the problem. In peculiar cases, the characteristic function is expressed by elementary functions. Graphs of the probability density function are presented for a few cases. The article is a continuation of the author's previous paper.

KEY WORDS: Langevin equation; quadratic noise; probability distribution; exactly solved problem; relaxation problem.

1. INTRODUCTION

In a recent article[1] I considered the following linear first-order stochastic differential equation (I use the notations of Ref. 1):

$$\dot{z}_t = cz_t + \mu(\lambda + y_t)^2 \tag{1.1}$$

where $z \in \mathbb{R}$, for $t = 0$, $z_t = z_0$, $c \in \mathbb{R}$, $\mu \in \mathbb{R}$, and y_t is the colored noise stochastic process defined by

$$\langle y_t \rangle = 0; \qquad \langle y_t y_s \rangle = (\gamma/\alpha) \exp(-\alpha\, |t-s|) \tag{1.2}$$

with γ and α fixed positive constants.

The process y_t is generated by a stochastic differential equation

$$dy_t = -\alpha y_t\, dt + (2\gamma)^{1/2}\, dW_t \tag{1.3}$$

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0022-4715/86/1000-0309$05.00/0

under the assumption that

$$\langle y_0 \rangle = 0; \qquad \langle y_0^2 \rangle = \gamma/\alpha \tag{1.4}$$

and W_t is the standard Wiener process.

The probability density for y_t has a Gaussian form

$$P_1(y, t) = (\alpha/2\pi\gamma)^{1/2} \exp(-\alpha y^2/2\gamma) \tag{1.5}$$

and does not depend on time.

Our aim is to obtain the one-dimensional probability distribution $P(z, t)$ for the process z_t in (1.1) for all $t > 0$ and with the initial condition

$$P(z, 0) = \delta(z - z_0) \tag{1.6}$$

The problem is less trivial than it looks. In a previous paper[1] I obtained the probability function $P(z, t)$ for the particular value of the parameters $c = -4\alpha$ and $\lambda = 0$. Here, I give an extension of the results in Ref. 1 for an arbitrary value of the parameter c and show that the case $\lambda \neq 0$ can also be analyzed.

The remainder of this paper is organized as follows. Section 2 reviews essential formulas and equations contained in Ref. 1. I present a set of differential equations, the solution of which is indispensable to the calculation of the characteristic function of the process z_t. The solution of these equations is presented in Section 3. In general, the solution is expressed in terms of the Bessel, Neumann, and Lommel functions. In Section 4, I present the characteristic function and consider several special cases in which the characteristic function can be expressed by elementary functions. I also show a few curves for the probability distribution function in the stationary case.

2. METHOD OF SOLUTION OF THE PROBLEM

The probability distribution $P(z, t)$ of the process z_t in (1.1) can be obtained from the characteristic function $C(\omega, t)$ via its exponential Fourier transform, namely

$$P(z, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega\, e^{-i\omega z} C(\omega, t) \tag{2.1}$$

and $C(\omega, t)$ can be expressed as[1]

$$C(\omega, t) = F[y_t \mid \omega, t] \exp[i\omega z_0 e^{ct} + i\omega\mu\lambda^2(e^{ct} - 1)/c] \tag{2.2}$$

The functional $F[y_t | \omega, t]$ is obtained from the following equality:

$$F[y_t | \omega, t] = F_1[y_t | \Omega = \omega\mu e^{ct}, t] \tag{2.3}$$

where

$$F_1[y_t | \Omega, t] = \int_{-\infty}^{+\infty} \Gamma(y, t)\, dy \tag{2.4}$$

and the "curtailed" functional $\Gamma(y, t)$ obeys the partial differential equation[2]

$$\frac{\partial \Gamma(y, t)}{\partial t} = \left[\alpha \frac{\partial}{\partial y} y + \gamma \frac{\partial^2}{\partial y^2} + i\Omega e^{-ct}(y^2 + 2\lambda y) \right] \Gamma(y, t) \tag{2.5}$$

with the initial condition[2]

$$\Gamma(y, 0) = P_1(y, 0) \tag{2.6}$$

Look for a solution of (2.5) in the form

$$\Gamma(y, t) = \exp[A(t)\, y^2 + B(t)\, y + C(t)] \tag{2.7}$$

Inserting the ansatz into (2.5), we find that the functions $A(t)$, $B(t)$, and $C(t)$ are solutions to the following problem:

$$\dot{A} = 4\gamma A^2 + 2\alpha A + i\Omega e^{-ct} \tag{2.8a}$$

$$\dot{B} = \alpha B + 4\gamma AB + 2i\lambda\Omega e^{-ct} \tag{2.8b}$$

$$\dot{C} = \alpha + 2\gamma A + \gamma B^2 \tag{2.8c}$$

From Eqs. (2.7), (2.6), and (1.5) it follows that

$$A(0) = -\alpha/2\gamma; \qquad B(0) = 0; \qquad C(0) = \tfrac{1}{2}\ln(\alpha/2\pi\gamma) \tag{2.9}$$

Now, the main problem reduces to solving the system of ordinary differential equations (2.8) with (2.9). In Ref. 1, I solved the Riccati equation (2.8a) only for $c = -4\alpha$. It turns out that the above Riccati equation can be solved for an arbitrary value of c. This is done in the next section.

3. SOLUTION OF DIFFERENTIAL EQUATIONS

First, we solve the Riccati equation (2.8a). By using the transformation

$$A(t) = -\dot{X}(t)/4\gamma X(t) \tag{3.1}$$

and the change of the independent variable

$$s = 4i\gamma\Omega e^{-ct} \tag{3.2}$$

one transforms the Riccati equation into a linear differential equation of the second-order

$$s\ddot{X}(s) + \left(1 + \frac{2\alpha}{c}\right)\dot{X}(s) + \frac{1}{c^2}X(s) = 0 \tag{3.3}$$

The above equation was obtained in Ref. 1, and I wrote that an explicit solution of this equation was known for two cases. I used Kamke's text,[3] in which an equation of the same form was contained, and suggested that these two cases exhausted all explicit solutions. Recently, I have succeeded in solving this equation without any restrictions on the parameters. For this purpose we must perform one more transformation, namely

$$X(s) = s^{-1/2}s^{-\alpha/c}Y(s) \tag{3.4}$$

This leads to

$$s^2\ddot{Y}(s) + \left(\frac{s}{c^2} + \frac{1}{4} - \frac{\alpha^2}{c^2}\right)Y(s) = 0 \tag{3.5}$$

A solution of this equation is known[4] (below I use the notations of Ref. 4) and reads

$$Y(s) = s^{1/2}Z_\nu\left(\frac{2}{c}s^{1/2}\right) \tag{3.6}$$

where

$$\nu = 2\alpha/c \tag{3.7}$$

and Z_ν stands for an arbitrary solution of Bessel's differential equation.

Taking into account Eqs. (3.1), (3.2), and (3.4), we can write the general solution in the form

$$X(t) = e^{\alpha t}[C_0 J_\nu(f_t) + M_\nu(f_t)] \tag{3.8}$$

where

$$f_t = f_0 \exp(-ct/2); \qquad f_0 = 4(i\gamma\Omega)^{1/2}/c \tag{3.9}$$

and

$$M_\nu = Y_\nu \qquad \text{or} \qquad M_\nu = J_{-\nu} \tag{3.10}$$

are the Neumann and Bessel functions, respectively.

The constant C_0 is determined from the initial condition (2.9) for $A(t)$ and reads

$$C_0 = -\frac{M_\nu(f_0) \pm g M_{\nu+1}(f_0)}{J_\nu(f_0) - g J_{\nu+1}(f_0)} \tag{3.11}$$

with

$$g = (i\gamma\Omega)^{1/2}/\alpha \tag{3.12}$$

In Eq. (3.11) and in all expressions below, the upper sign refers to $M_\nu = J_{-\nu}$ and the lower sign refers to $M_\nu = Y_\nu$.

Using recurrence relations for the Bessel and Neumann functions,[4] from Eq. (3.8) we have

$$\dot{X}(t) = 2\alpha g \exp[(2\alpha - c)\, t/2][C_0 J_{\nu+1}(f_t) \mp M_{\nu+1}(f_t)] \tag{3.13}$$

Let us now solve the remaining equations (2.8). Substituting (3.1) into (2.8b) and taking into account (2.9), we find the solution for $B(t)$,

$$\begin{aligned} B(t) = {} & -(\lambda c/4\gamma)[C_0 J_\nu(f_t) + M_\nu(f_t)]^{-1} \\ & \times \int_{f_0}^{f_t} x[C_0 J_\nu(x) + M_\nu(x)]\, dx \end{aligned} \tag{3.14}$$

The integral in Eq. (3.14) can be calculated[4]

$$\int x J_\nu(x)\, dx = \nu x J_\nu(x)\, S_{0,\nu-1}(x) - x J_{\nu-1}(x)\, S_{1,\nu}(x) \tag{3.15}$$

The same formula holds for the Neumann function and $S_{\mu,\nu}$ stands for the Lommel function.

Using Eq. (3.15) and the recurrence relations for the Lommel and Bessel functions, one obtains the solution (3.14) explicitly as

$$\begin{aligned} (2\gamma/\lambda)\, B(t) = {} & \alpha f_t S_{0,\nu+1}(f_t) - \dot{X}(t)\, S_{1,\nu}(f_t)/X(t) \\ & + \alpha g f_0 e^{\alpha t} S_{0,\nu-1}(f_0)\, W/X(t) \end{aligned} \tag{3.16}$$

where $W = W\{J_\nu(f_0), M_\nu(f_0)\}$ is the Wronskian of the functions $J_\nu(f_0)$ and $M_\nu(f_0)$,

$$W\{J_\nu(x), J_{-\nu}(x)\} = (2/\pi x) \sin \pi(\nu+1) \tag{3.17a}$$

$$W\{J_\nu(x), Y_\nu(x)\} = 2/\pi x \tag{3.17b}$$

Equation (2.8c) has the following solution:

$$C(t) = \alpha t + \frac{1}{2} \ln[\alpha X(0)/2\pi\gamma X(t)] + \gamma \int_0^t ds\, B^2(s) \tag{3.18}$$

Equations (3.1) with (3.8) and (3.13) as well as Eqs. (3.16) and (3.18) determine the "curtailed" functional $\Gamma(y, t)$, (2.7), in a unique manner, and from Eq. (2.4) we are able to find the functional $F_1[y_t | \Omega, t]$.

4. FINAL SOLUTIONS AND SOME SPECIAL CASES

The formula for the functional $F_1[y_t | \Omega, t]$ can be presented in the form

$$F_1[y_t | \Omega, t] = [2\alpha X(0)/\dot{X}(t)]^{1/2} \times \exp\left[\alpha t + \gamma X(t)\, B^2(t)/\dot{X}(t) + \gamma \int_0^t B^2(s)\, ds\right] \tag{4.1}$$

The dependence of $F_1[y_t | \Omega, t]$ upon Ω is hidden in the parameters f_0 [Eq. (3.9)] and g [Eq. (3.12)]. The expression (4.1) holds for arbitrary values of parameters $c, \mu, \lambda \in \mathbb{R}$ and $\alpha, \gamma \in \mathbb{R}^+$. A compact formula for the characteristic function $C(\omega, t)$ can be obtained for the case $\lambda = 0$ and becomes

$$C(\omega, t) = (g_\omega)^{-1/2} \exp\left[\frac{(2\alpha - c)\, t}{4} + i\omega z_0 e^{ct}\right] \times \left[\frac{\mp W\{J_\nu(f_\omega), M_\nu(f_\omega)\}}{J_{\nu+1}(f_\omega)\, M_{\nu-1}(f_\omega(t)) - M_{\nu+1}(f_\omega)\, J_{\nu-1}(f_\omega(t))}\right]^{1/2} \tag{4.2}$$

where

$$f_\omega(t) = f_\omega \exp(ct/2) \tag{4.3a}$$

$$f_\omega = (4\alpha/c)\, r_0 \sqrt{\omega} \tag{4.3b}$$

$$g_\omega = r_0 \sqrt{\omega} \tag{4.3c}$$

and

$$r_0 = (i\gamma\mu)^{1/2}/\alpha \tag{4.3d}$$

The characteristic function (4.2) can be expressed by elementary functions for the half odd integer order

$$\nu = (2n+1)/2; \qquad n = 0, \pm 1, \pm 2, \ldots \tag{4.4}$$

First, let us consider the positive order, $\nu > 0$.

A. In the simplest case, $n=0$, $\nu=1/2$, and $c=4\alpha>0$. For this case we have

$$F[y_t|\omega, t] = \{\cos[g_\omega(e^{2\alpha t}-1)] - g_\omega \sin[g_\omega(e^{2\alpha t}-1)]\}^{-1/2} \quad (4.5)$$

It can be shown that although $P(z, t)$ exists for all finite time $t>0$, the stationary distribution satisfies

$$P_{\text{st}}(z) = \lim_{t\to\infty} P(z, t) = 0 \quad (4.6)$$

This is correct for all $c>0$ and it is related to unstable solutions of Eq. (1.1). The stationary density exists only for $c<0$. The case $c<0$ refers to relaxation and

$$P_{\text{st}}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, e^{-i\omega z} F_{\text{st}}[y_t|\omega] \quad (4.7)$$

where

$$F_{\text{st}}[y_t|\omega] = \lim_{t\to\infty} F[y_t|\omega, t] \quad (4.8)$$

and, as is seen from Eq. (2.2), $P_{\text{st}}(z)$ does not depend on the initial state z_0 of the process (an ergodic process).

B. For the first negative order $n=-1$, $\nu=-1/2$, and $c=-4\alpha$. This case was considered in Ref. 1.

C. For $\nu=-3/2$, $c=-4\alpha/3$ and

$$\begin{aligned} F[y_t|\omega, t] &= (3g_\omega)^{1/2}\{(1-3g_\omega^2 e^{-4\alpha t/3})\sin[3g_\omega(1-e^{-2\alpha t/3})] \\ &\quad + 3g_\omega e^{-2\alpha t/3}\cos[3g_\omega(1-e^{-2\alpha t/3})]\}^{-1/2} \end{aligned} \quad (4.9)$$

and

$$F_{\text{st}}[y_t|\omega] = [3g_\omega/\sin(3g_\omega)]^{1/2} \quad (4.10)$$

D. For $\nu=-5/2$, $c=-4\alpha/5$, the expression for $F[y_t|\omega, t]$ is quite lengthy, but the stationary form reads

$$F_{\text{st}}[y_t|\omega] = \{(5g_\omega)^3/[3\sin(5g_\omega) - 15g_\omega\cos(5g_\omega)]\}^{1/2} \quad (4.11)$$

A detailed analysis of the results obtained will be presented in a separate paper. The distribution function $P(z, t)$ of the process of interest is calculated from Eq. (2.1) by the numerical integration of the many-valued functions [as, e.g., (4.9)]. In order to obtain insight into the shape of the probability distribution, consider Fig. 1, which shows the stationary probability density $P_{st}(z)$ for case B, $\nu = -1/2$,

$$P_{st}(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \frac{e^{-i\omega z}}{(\cos g_{\omega})^{1/2}} \tag{4.12}$$

and a new parameter $\sigma \in \mathbb{R}$,

$$\gamma = \alpha^2 \sigma^2 / 2 \tag{4.13}$$

has been introduced. It will facilitate the study of the problem of the white noise limit $\alpha \to \infty$ in Eq. (1.1) (see Section 6 in Ref. 1). In the case of Eq. (4.13), we have only one free parameter $\sigma^2 \mu$ in (4.12). The characteristic feature of the probability function (4.12) is its vanishing for $z < 0$ when $\mu > 0$ and vice versa. In other words, $P_{st}(z)$ is proportional to the

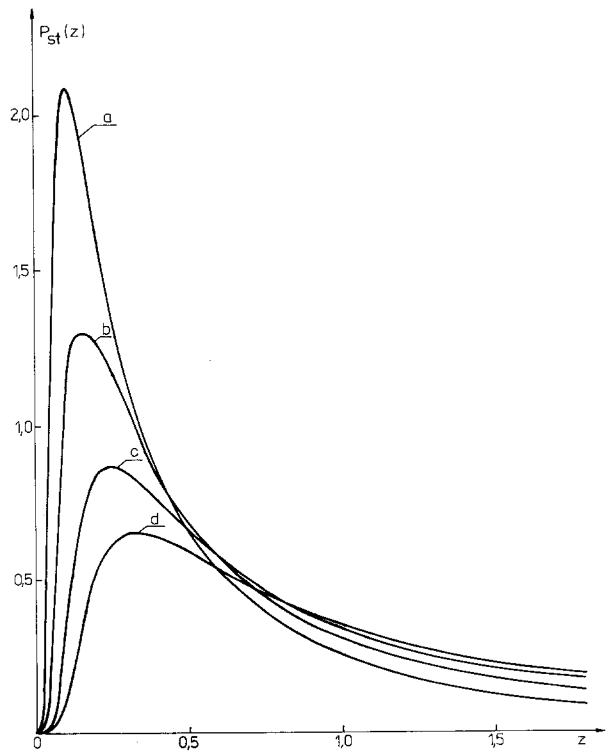

Fig. 1. Shape of the probability distribution $P_{st}(z)$ for $c = -4\alpha$ the following values of $\sigma^2\mu$: (a) 5, (b) 8, (c) 12, (d) 16.

Heaviside function $\theta(z \operatorname{sign} \mu)$. This is due to the fact that singularities of the integrand lie on the negative half of the imaginary axis for $\mu > 0$. For $\mu < 0$, the probability density can be obtained form Fig. 1 by reflection with regard to the vertical axis.

In closing, I note that the remarks contained in Section 6 in Ref. 1, which concern the white noise limit, are correct for all negative fixed values of the parameter v because in this case c is proportional to the parameter α.

ACKNOWLEDGMENTS

I appreciate the great assistance with numerical calculations given by A. Gryłka. I am also indebted to Dr. E. Piotrowski for discussions and remarks. This work was supported in part by the Polish Academy of Sciences under contract MR I-9.

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